Dynkin diagrams and crepant resolutions of quotient singularities

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Introduction

Let V be a complex vector space, and assume that a finite group G acts on V by linear transformations. Consider the quotient V/G. If $\dim V = 1$, the quotient V/G is a smooth algebraic variety. If $\dim V \geq 2$, it may still happen that the quotient V/G is smooth. However, usually the algebraic variety V/G is singular. In this case it is natural to look for good resolutions of singularities $X \to V/G$ of the quotient V/G.

There exists a well-known situation when there indeed exists such a resolution: the so-called *McKay correspondence*. This is the case when the vector space $V = \mathbb{C}^2$ is of dimension $\dim V = 2$, and the finite subgroup $G \subset SL(2,\mathbb{C})$ preserves the standard symplectic form on V. For such a quotient V/G, one can construct a canonical smooth resolution $X \to V/G$ which has many good properties. One of these properties is that the smooth algebraic variety X has trivial canonical bundle. Such resolutions are called *crepant*.

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In their recent paper [IN], Y. Ito and H. Nakajima studied possible generalizations of the McKay correspondence to higher dimensions. One of the questions that they asked was the following: for which pairs $\langle V, G \rangle$ the quotient V/G admits a crepant smooth resolution $X \to V/G$?

Aside from the McKay correspondence, the best-known example of such a pair is the $2n$-dimensional complex vector space $V = \mathbb{C}^{2n}$ and the symmmetric group $G = S_n$ on n letters that acts on V by transpositions, separately on odd and even coordinates. In this case, a crepant resolution X is the Hilbert scheme of n points on $\mathbb{C}^2$ (see [N2]).

We note that both in the case of the McKay correspondence and in the case of the Hilbert scheme, the complex vector space V carries a symplectic form preserved by the group G. This suggests that the natural first choice for a quotient that admits a crepant resolution would be the quotient V/G of a vector space V which carries a G-invariant symplectic form.

Such symplectic vector spaces are, of course, very common. A simple way to construct them is by "doubling" representations of finite groups: for an arbitrary vector space V_o equipped with a linear action of a finite group G, the sum $V = V_o \oplus V_o^*$ of the vector space V_o with its dual V_o^* carries both a natural G-action and a canonical G-invariant symplectic form.

In this paper, we study smooth crepant resolutions of quotients V/G, where $V = V_o \oplus V_o^*$ is a symplectic vector space obatined by doubling a representation V_o of a finite group G. The main result of this paper, Theorem 1.7, claims that if for $V = V_o \oplus V_o^*$ the quotient V/G admits a smooth crepant resolution, then the action of the group G on the vector space V_o is generated by (complex) reflections.

Finite groups $G \subset \operatorname{Aut} V_o$ of automorphisms of a complex vector space V_o which are generated by reflections have been an object of much study, and there exists a complete classification of pairs $\langle V_o, G \rangle$ of this type. The best-known part of this classification concerns subgroups $G \subset \operatorname{Aut} V_o$ which preserve a rational lattice $V_{\mathbb{Q}} \subset V_o$. In this case G is generated by reflections if and only if it is a product of Weyl groups associated to Dynkin diagrams of finite type (see [B, Ch. VI, §2, p. 5, Proposition 9]).

Of all the subgroups $G \subset \operatorname{Aut}(\mathbb{C}^2)$ studied in the McKay correspondence, only the simplest ones preserve the decompostion $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$, thus falling under the assumptions of our main theorem. These are the cyclic groups $G = \mathbb{Z}/n\mathbb{Z}$. From the point of view of the theory of groups generated by reflections, this is the trivial case. The case of the Hilbert scheme is more interesting. It also falls under our assumptions, and corresponds to the Weyl group of the type A_n.

Among the quotients by the Weyl groups of other types, a crepant resolution has been constructed for the type C_n by A. Kuznecov ([K], see also [KV]). For the rest of the Dynkin diagrams the question posed in [IN] seems to be open.

Aside from our main theorem, we also obtain some results on the structure of smooth crepant resolutions of quotient sungularities V/G, and we prove some additional facts. Among these, we would like to mention Theorem 1.3. It claims, more or less, that if a connected algebraic group H acts on the space V, and if the H-action commutes with the G-action, then for any smooth crepant resolution $\pi : \widetilde{X} \to X$ the induced H-action on the quotient $X = V/G$ lifts to an H-action on the variety $\widetilde{X}$. We refer the reader to Section 1 for the precise statement.

For this result, we do not need the assumption $V = V_o \oplus V_o^*$. It holds for an arbitrary symplectic vector space V and a finite subgroup $G \subset \operatorname{Aut} V$ which preserves the symplectic form. In fact, even the symplectic form is not necessary – it suffices to assume that the group G preserves a volume form on the space V (see Remark 3.7).

Another result which we would like to mention here is the uniqueness statement Theorem 1.9. It claims that if any two reflections in the subgroup $G \subset \operatorname{Aut} V_o$ are conjugate within G, then there exists at most one smooth crepant resolution of the quotient V/G. When $G \subset \operatorname{Aut} V_o$ is a Weyl group, this condition is satisfied if and only if the corresponding Dynkin diagram is simply laced.

Unfortunately, for technical reasons we need to impose an additional restriction: in Theorem 1.9 we require that the complex vector space V_o admits a real structures invariant under G. Among other things, this implies that $V_o \cong V_o^*$ as representations of the group G.

Let us now give a brief outline of the contents of the paper.

- Section 1 contains the precise description of the general setup, the definitions, and the statements of all the results proved in the paper.

- A very short Section 2 gives an overview of the simplest possible case $V = \mathbb{C}^2$, $G = \{\pm 1\}$.

- Section 3 is devoted to some general technical results – in particular, we prove that under our assumptions, for every smooth crepant resolution $X \to V/G$ the variety X is holomorphically symplectic. The reader is advised to skip this section at first reading.

- In Section 4, we introduce a certain stratification of the quotient V/G.

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- This stratification is then used in Section 5 to study in some detail the geometry of an arbitrary smooth crepant resolution $X \to V/G$. In particular, we compute the rational Picard group $\mathrm{Pic}(X) \otimes \mathbb{Q}$. After that we are able to prove most of our results – in fact all of them, except for the main Theorem 1.7.

- In Section 6 we introduce, following [N1], a certain canonical action of the group $\mathbb{C}^*$ on an arbitrary smooth crepant resolution $X \to V/G$. Using this action, we then prove Theorem 1.7.

To finish the introduction, I would like mention that while this paper was in preparation, a stronger version of Theorem 1.7 was proved by M. Verbitsky. This will be the subject of his upcoming paper [V2]. His proof is different from the one presented here; it is much simpler and it does not use most of our technical results. Thus the last section of this paper is perhaps obsolete. I have decided to keep it anyway, since the $\mathbb{C}^*$-action which is constructed and studied there may be of independent interest.

1   Preliminaries

Let V be a complex vector space of dimension $2n$ equipped with a non-degenerate symplectic form Ω_V. Assume given a finite group $G \subset \mathrm{Aut}\, V$ of automorphisms of the vector space V which preserves the symplectic form. The quotient V/G is then naturally a singular affine algebraic variety over $\mathbb{C}$. In this paper we will interested in smooth resolutions of the variety V/G.

To fix terminology, we introduce the following.

Definition 1.1. A map $\pi : \widetilde{X} \to X$ is called a *resolution* of an irreducible algebraic variety X if the algebraic variety $\widetilde{X}$ is irreducible, and the map π is generically one-to-one. A resolution $\pi : \widetilde{X} \to X$ is called *smooth* if the algebraic variety $\widetilde{X}$ is smooth, and *projective* if the map $\pi : \widetilde{X} \to X$ is projective.

We will only be interested in a special class of resolutions, the so-called *crepant* ones. The general definition is as follows. Let $\pi : \widetilde{X} \to X$ be a

resolution of an irreducible algebraic variety X. Assume that both X and $\widetilde{X}$ are normal and admit canonical bundles K_X and $K_{\widetilde{X}}$[1]. Let $U \subset X$ be a non-singular open dense subset such that $\pi : \widetilde{U} = \pi^{-1}(U) \to U$ is one-to-one.

Definition 1.2. The resolution $\pi : \widetilde{X} \to X$ is called *crepant* if the canonical isomorphism

$$\pi^* K_U \cong K_{\widetilde{U}},$$

over $U = \pi^{-1}(U) \subset \widetilde{X}$ extends to a bundle isomorphism $\pi^* K_X \cong K_{\widetilde{X}}$ over the whole $\widetilde{X}$.

The quotient variety $X = V/G$ is normal and irreducible. Moreover, it obviously admits a canonical bundle K_X – namely, the trivial one. Therefore for every crepant resolution $\pi : \widetilde{X} \to X$ the canonical bundle $K_{\widetilde{X}}$ is also trivial.

For projective resolutions, this necessary condition is in fact sufficient: a projective resolution $\pi : \widetilde{X} \to X$ is crepant if the canonical bundle $K_{\widetilde{X}}$ is trivial. Indeed, in this case a map

$$\pi^* K_X \cong \mathcal{O}_{\widetilde{X}} \to K_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$$

is simply a function on $\widetilde{X}$. We are given a function f without zeroes on $\widetilde{U} = \pi^{-1}(U) \subset \widetilde{X}$, and we have to prove that it extends to a function without zeroes on the whole $\widetilde{X}$. It suffices to extend $\pi^{-1} \circ f : U \to \mathbb{C}$ to a function f without zeroes on X and take $\pi^* f$. But since X is normal and the map $\pi : \widetilde{X} \to X$ is dominant, we can assume that the complement $X \backslash U \subset X$ to the open non-singular subset $U \subset X$ is of codimension $\operatorname{codim} X \setminus U \geq 2$. Again using the fact that X is normal, we see that every function f without zeroes on U extends to a function without zeroes on the whole X.

In this paper we consider a quotient $X = V/G$ of a symplectic vector space V which admits a smooth projective crepant resolution $\pi : \widetilde{X} \to X$, and we prove assorted results on the geometry of the quotient X which hold under varying linear-algebraic assumptions on the pair $\langle V, G \rangle$. Our first result holds for every symplectic vector space.

Theorem 1.3. *Let $X = V/G$ be the quotient of a symplectic vector space V by a symplectic action of a finite group $G \subset \operatorname{Aut}(V)$. Assume given a smooth projective crepant resolution $\pi : \widetilde{X} \to X$.*

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[1]An algebraic variety X is said to admit a canonical bundle if the canonical bundle of the non-singular part of X extends to a line bundle on the whole X. If X is normal, then such an extension is unique

(i) *For every G-invariant vector field $\mathfrak{t}_V$ on V, the induced vector field $\mathfrak{t}_0$ on the non-singular part $X_0 \subset X = V/G$ lifts to a vector field $\mathfrak{t}$ on the whole smooth algebraic variety $\widetilde{X}$.*

(ii) *If a connected algebraic group H acts on the algebraic variety V, and if the H-action commutes with the G-action, then the induced H-action on the quotient $X = V/G$ lifts to an H-action on the variety $\widetilde{X}$.*

(iii) *The same is true for an arbitrary, not necessarily connected algebraic group H, which acts on the vector space V by linear transformations.*

Our second general result is somewhat technical, but it might be useful in applications. To formulate this result, consider the formal completion $\widehat{X}$ of a quotient variety $X = V/G$ at $0 \in V/G$. Say that a formal scheme $\mathfrak{X}$ equipped with a projective map $\pi : \mathfrak{X} \to \widehat{X}$ is a smooth crepant resolution of the completion $\widehat{X}$ if $\mathfrak{X}$ is smooth and the canonical bundle $K_\mathfrak{X}$ is trivial.

Theorem 1.4. *Assume given a smooth projective crepant resolution $\pi : \mathfrak{X} \to \widehat{X}$ of the completion $\widehat{X}$. Then there exists a unique smooth projective crepant resolution $\pi : \widetilde{X} \to X$ which gives $\pi : \mathfrak{X} \to \widehat{X}$ after completing along the fiber $\pi^{-1}(0) \subset \widetilde{X}$.*

Remark 1.5. By virtue of Grothendieck's algebraization theorem [EGA, Théorème 5.4.5], one can replace here a formal scheme $\mathfrak{X}$ with a usual scheme projective over $\widehat{X}$.

Our final result which holds for every symplectic vector space V is the following.

Theorem 1.6. *Assume that the quotient $X = V/G$ of a symplectic vector space V by a symplectic action of a finite group $G \subset \mathrm{Aut}(V)$ admits a smooth projective crepant resolution. Let $v \in V$ be an arbitrary vector, let $G_v \subset G$ be the stabilizer of the vector $v \in V$, and let $V' \subset V$ be the unique G_v-invariant complement to the subspace $V^{G_v} \subset V$ of vectors fixed by G_v.*

Then the quotient $X_v = V'/G_v$ also admits a smooth projective crepant resolution.

The main result of the paper holds only under an additional assumption. To formulate this result, recall that a finite-order automorphism $g : V \to V$ of a vector space V is called a *complex reflection* if the subspace $V^g \subset V$ of g-invariants has codimension exactly 1.

Theorem 1.7. *Let V be a symplectic vector space, and let $G \subset \mathrm{Aut}(V)$ be a finite group of symplectic automorphims of the vector space V. Assume that the symplectic vector space V admits a G-invariant Lagrangian subspace $V_o \subset V$.*

If the quotient $X = V/G$ admits a smooth projective crepant resolution $\pi : \widetilde{X} \to X$, then the subgroup $G \subset \mathrm{Aut}(V_o)$ is generated by complex reflections.

Note that for an arbitrary representation V_o of a finite group G, the vector space $V = V_o \oplus V_o^*$ is symplectic and satisfies the conditions of Theorem 1.7.

Theorem 1.7 is analogous to the following classic fact.

Theorem 1.8 ([B, Ch. V, §5, Theorem 4]). *The quotient V/G of a complex vector space V by a finite subgroup $G \subset \mathrm{Aut}(V)$ is smooth if and only if the subgroup $G \subset \mathrm{Aut}(V)$ is generated by complex reflections.* $\square$

In fact, our proof of Theorem 1.7 uses this fact directly. Note, however, that we do not claim the converse to Theorem 1.7. At present, we do not know for which of the subgroups $G \subset \mathrm{Aut}(V_o)$ generated by complex reflections the quotient V/G admits a smooth crepant resolution. We hope to return to this in a later paper.

In the course of proving Theorem 1.7, we also establish the following uniqueness statement, which might be of independent interest. Unfortunately, we can only prove this result under an even stronger additional assumption.

Theorem 1.9. *In the notation of Theorem 1.7, assume that the complex vector space V_o admits a real structure preserved by the group G. Assume also that the reflections in G form a single conjugacy class.*

Then every two smooth projective crepant resolutions $\pi_1 : \widetilde{X}_1 \to X$, $\pi_2 : \widetilde{X}_2 \to X$ of the quotient V/G are canonically isomorphic.

In fact, the smoothness assumption on $\widetilde{X}_1$, $\widetilde{X}_2$ imposed in this theorem can be weakened (see Remark 5.8).

2 The case of dimension 1

We begin with the simple and well-known case $\mathsf{dim}\, V = 2$. In this case Theorem 1.6 and Theorem 1.7 are trivial. On the other hand, the proofs

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of Theorem 1.3 and Theorem 1.4 are essentially the same as in the general case. We postpone these till Section 5. In this section we concentrate on the uniqueness result, Theorem 1.9.

Under the additional assumptions of Theorem 1.9, the group G preserves a 1-dimensional Lagrangian subspace $V_o \subset V$ and a real structure on V_o. Therefore G necessarily consists of multiplications by ± 1. The affine quotient variety $X = V/\{\pm 1\}$ is the singular quadric in $\mathbb{C}[u, v, w]$ given by the equation $uw = v^2$, and the quotient map $V = \mathbb{C}[x, y] \to X$ sends u to x^2, v to xy and w to y^2.

A classic crepant resolution for X is obtained by blowing up $0 \subset X$. It coincides with the total space of the cotangent bundle to the complex projective line $\mathbb{C}P^1$.

We will now prove Theorem 1.9 for this simple case. Assume given another smooth projective crepant resolution $\pi : \widetilde{X} \to X$. By definition the map π is one-to-one over a non-singular open subset $X_0 \subset X$, so that $\pi : \widetilde{X}_0 = \pi^{-1}(X_0) \to X_0$ is an isomorphism. Since $X = V/G$ is normal, we can assume that the complement $X \setminus X_0$ is of codimension $\mathsf{codim}\, X \setminus X_0 > 1$. Moreover, we can assume that X_0 does not contain $0 \in X$, so that the quotient map $V \to X$ is étale over $X_0 \subset X$.

The G-invariant symplectic form Ω_V on V induces a non-degenerate symplectic form Ω on $X_0 \subset X$. But $\mathsf{dim}\, X = 2$. Therefore a symplectic form on X_0 is a section of the canonical bundle K_X. Since the resolution $\pi : \widetilde{X} \to X$ is crepant, the pullback $\pi^*\Omega$ extends to a non-degenerate symplectic form Ω on the whole smooth algebraic variety $\widetilde{X}$. The 2-form Ω on $\widetilde{X}$ induces a canonical isomorphism

$$ \omega : H^0(\widetilde{X}, \mathcal{T}) \xrightarrow{\sim} H^0(\widetilde{X}, \Omega^1) $$

between the space of global 1-forms on $\widetilde{X}$ and the space of global vector fields on $\widetilde{X}$.

The standard action of the group $SL(2, \mathbb{C})$ on the vector space $V = \mathbb{C}^2$ commutes with the action of the group G. Therefore it descends to an action on the space X. The corresponding infinitesemal action is generated by three vector fields E, F, H on $X_0 \cong \widetilde{X}_0$ which generate the Lie algebra $\mathfrak{sl}(2)$. Over X_0, the isomorphism ω sends these three vector fields to the differential forms du, dv and dw.

Since these forms obviously extend to the whole $\widetilde{X}$, we conclude that the vector fields E, F and H also extend to the whole $\widetilde{X}$.

By assumption the map $\pi : \widetilde{X} \to X$ is projective. Therefore the variety $\widetilde{X}$ carries a line bundle L which is very ample for the map π, so that we

have

$$\widetilde{X} = \operatorname{Proj} \bigoplus_{k \geq 0} \pi_* L^k.$$

Let $j : X_0 \hookrightarrow X$ be the embedding map of the open subset $X_0 = X \setminus \{0\} \subset X$. Since the codimension $\operatorname{codim} X \setminus X_0$ is greater than 1, the Picard group of the open subset $\widetilde{X}_0 \cong X_0 \subset X$ coincides with the Picard group of the whole variety X, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Replacing L with its positive power, we can assume that $L|_{\widetilde{X}_0}$ is trivial. Any isomorphism $\pi_* L \xrightarrow{\sim} \mathcal{O}_X$ over X_0 extends to an embedding

$$\pi_* L \hookrightarrow j_* \mathcal{O}_X \cong \mathcal{O}_X.$$

This means that the sheaf $\pi_* L$ is in fact a sheaf of ideals in $\mathcal{O}_X$, and the resolution $\pi : \widetilde{X} \to X$ is the blowup of the variety X in this sheaf of ideals.

Since the $\mathfrak{sl}(2)$-action on X lifts to an action on $\widetilde{X}$, the sheaf of ideals $\pi_* L \subset \mathcal{O}_X$ must be $\mathfrak{sl}(2)$-invariant. Equivalently, the ideal of global sections $H^0(X, \pi_* L) \subset H^0(\mathcal{O}_X)$ must be $\mathfrak{sl}(2)$ invariant. But the only $\mathfrak{sl}(2)$-invariant ideals in the algebra $H^0(\mathcal{O}_X) = \mathbb{C}[u, v, w]/(uw - v^2)$ are the powers of the maximal ideal corresponding to the point $0 \in X$. Consequently, $\widetilde{X}$ must be isomorphic to the blow-up of the point $0 \in X$.

This argument proves Theorem 1.9 in the case $V = \mathbb{C}^2$. We sum up the results for this simple situation in the following statement.

Proposition 2.1. *Every projective smooth crepant resolution $\pi : \widetilde{X} \to X$ is isomorphic to the blow-up of the point $0 \in X$. Moreover, the Picard group of line bundles on $\widetilde{X}$ is $\mathbb{Z}$, and for the ample line bundle corresponding to $2k \in \mathbb{Z}$, $k \geq 1$ we have a canonical isomorphism*

$$\pi_* L \cong \mathfrak{m}^k,$$

where $\mathfrak{m} \subset \mathcal{O}_X$ is the sheaf of ideals of the point $0 \in X$. $\qquad\square$

Note that everything in the proof carries over literally to the formal situation of Theorem 1.4. Therefore the same statements hold for projective smooth crepant resolutions of the formal completion $\widehat{X}$ of X at $0 \subset X$.

3 The holomorphic symplectic form

In higher dimensions the canonical bundle is no longer the bundle of 2-forms. Still, it turns out that for every symplectic vector space V and for

every finite subgroup $G \subset \operatorname{Aut} V$ which preserves the symplectic form, every crepant resolution of the quotient singularity $X = V/G$ carries a canonical holomorphic symplectic form Ω. The form Ω, which will be very important for all our constructions, is described in this section.

Let V be a complex vector space equipped with a non-degenerate symplectic form Ω_V. Assume given a finite group $G \subset \operatorname{Aut}(V)$ of automorphisms of the vector space V which preserves the form Ω_V. Let $X = V/G$ be the quotient variety. The quotient map $\sigma : V \to X = V/G$ is étale over a smooth open dense subset $X_0 \subset X$, moreover, it is a Galois covering with Galois group G. Since the form Ω is G-invariant, it defines a non-degenerate holomorphic 2-form $\Omega \in H^0(X_0, \Omega^2(X_0))$ over the subset $X_0 \subset X$. Moreover, for every smooth resolution $\pi : \widetilde{X} \to X$ the pull-back $\pi^*\Omega$ defines a holomorphic 2-form over the preimage $\widetilde{X}_0 = \pi^{-1}(X_0) \subset \widetilde{X}$.

Definition 3.1. A smooth resolution $\pi : \widetilde{X} \to X$ is called *symplectic* if the canonical 2-form $\pi^*\Omega$ on the open subset $\widetilde{X}_0 \subset \widetilde{X}$ extends to a non-degenerate symplectic form on the whole smooth variety $\widetilde{X}$.

Note that since we only consider irreducible resolutions $\widetilde{X}$, the open subset $\widetilde{X}_0 \subset \widetilde{X}$ is always dense, and such an extension Ω is unique.

The n-th power Ω^n of a non-degenerate symplectic 2-form on an n-dimensional smooth algebraic variety Y gives a section of the canonical bundle K_Y without zeroes. Therefore every symplectic resolution $\pi : \widetilde{X} \to X$ is crepant.

Our first result claims that the converse is also true. Namely, let $\pi : \widetilde{X} \to X$ be a smooth crepant resolution. Then we have the following.

Proposition 3.2. *The pull-back $\pi^*\Omega$ of the holomorphic symplectic form Ω on $X_0 \subset X$ extends to a non-degenerate symplectic form on the whole smooth variety $\widetilde{X}$.*

Proof.[2] The form $\pi^*\Omega$ is defined on the open subset $\widetilde{X}_0 = \pi^{-1}(X_0) \subset \widetilde{X}$. A 2-form on an n-dimensional smooth algebraic variety is non-degenerate if and only if the n-th power Ω^n has no zeroes. Since the resolution $\pi : \widetilde{X} \to X$ is crepant, the n-th power $\pi^*\Omega^n$ extends to the whole of $\widetilde{X}$ and has no zeroes. Therefore it suffices to prove that the form $\pi^*\Omega$ extends to a form Ω on the whole of $\widetilde{X}$. Every such extension will be automatically non-degenerate.

We will prove slightly more.

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[2]This proof, as in fact everything in the paper, works over an arbitrary algebraically closed field of characteristic 0. For a simpler proof which uses holomorphic geometry, see [V2] or [Beau].

Lemma 3.3. *Let V be an arbitrary connected smooth affine complex algebraic variety equipped with an action of a finite group $G \subset \mathrm{Aut}(V)$. Let $\pi : \widetilde{X} \to X$ be an arbitrary smooth resolution of the quotient variety $X = V/G$. Denote by $\sigma : V \to X$ the quotient map.*

Assume given a k-form $\alpha \in H^0(X_0, \Omega^k(X))$ on a smooth dense open subset $X_0 \subset X$, and assume that the pullback $\sigma^*\alpha$ extends to a form $\alpha_V \in H^0(V, \Omega^k(V))$ on the whole smooth algebraic variety V.

Then the pullback $\pi^*\alpha$ extends to a form $\alpha \in H^0(\widetilde{X}, \Omega^k(\widetilde{X}))$ on the whole smooth algebraic variety $\widetilde{X}$.

Proof. Since the variety $\widetilde{X}$ is smooth, it suffices to show that $\pi^*\alpha$ extends to the complement to a closed subvariety $Z \subset \widetilde{X}$ of codimension ≥ 2. In other words, we have to show that for every subvariety $Z \subset \widetilde{X} \setminus \widetilde{X}_0$ of codimension $\mathsf{codim}\, Z = 1$, the form α extends to an open neighborhood of a generic point $x \in Z$.

To do this, assume given an arbitrary (non-closed) point $x \in \widetilde{X} \setminus \widetilde{X}_0$ with the residue field K_x of transcendental dimension $\dim \widetilde{X} - 1$ over $\mathbb{C}$. We have to prove that the form $\pi^*\alpha$ extends to an open neighborhood of the point x.

Let $\mathcal{O}_{\widetilde{X},x}$ be the local ring of functions at x. Since $\widetilde{X}$ is smooth, the ring $\mathcal{O}_{\widetilde{X},x}$ is a regular discrete valuation ring and an algebra over the field K_x. Consider the subset

$$U = \mathrm{Spec}\, \mathcal{O}_{\widetilde{X},x} \subset \widetilde{X}.$$

The scheme U is a smooth open curve over the field K. It has two points: the special point $x \in U$, and the generic point $\eta \in U$, which coincides with the generic point of the irreducible variety $\widetilde{X}$. It suffices to prove that the form $\pi^*\alpha$ extends from the generic point $\eta \in U$ to the whole scheme U.

Since the map $\pi : \widetilde{X} \to X$ is generically one-to-one, it induces an isomorphism between the generic point $\eta \in U \subset \widetilde{X}$ and the generic point $\pi(\eta) \in X$. The quotient map $\sigma : V \to X$ is a Galois covering over $\pi(\eta) \subset X$ with Galois group G.

Let $\widetilde{U}$ be the normalization of the curve U in the field of functions $K(V)$. This is a curve over the residue field K_x.

The curve $\widetilde{U}$ is normal, therefore it is smooth. Since the curve U is also smooth, the canonical projection $\tau : \widetilde{U} \to U$ is flat. The group G acts naturally on $\widetilde{U}$, and the map $\tau : \widetilde{U} \to U$ induces an isomorphism $U \cong \widetilde{U}/G$.

We have a commutative square

$$
\begin{array}{ccc}
\widetilde{U} & \stackrel{\tau}{\longrightarrow} & U \\
\pi \downarrow & & \downarrow \pi \\
V & \stackrel{\sigma}{\longrightarrow} & X
\end{array}
$$

The pullback $\tau^*\pi^*\alpha$ is by definition a G-invariant k-form on the generic subset $\tau^{-1}(\eta) \subset \widetilde{U}$. But this form coincides with the pullback $\pi^*\sigma^*\alpha$, and the pullback $\sigma^*\alpha$ by assumption extends to form α_V on the whole V. We conclude that the pullback $\tau^*\pi^*\alpha$ extends to a G-invariant k-form α on the whole curve $\widetilde{U}$.

To finish the proof, it remains to apply the following simple algebraic fact.

Claim 3.4. *Let k be a field of finite degree of transcendence over $\mathbb{C}$, let K_1 be a Galois extension of the local field $K = k((x))$ with Galois group G, and let O_1 be the integral closure of the integer ring $O = k[[x]]$ in K_1. For every integer n, the have a canonical isomorphism of O-modules*

$$
\Omega^n(O/\mathbb{C}) \cong \Omega^n(O_1/\mathbb{C})^G.
$$

(Here Ω^n is understood as the n-th exterior power of the (flat) module Ω^1 of Kähler differentials.)

Proof. If the extension K_1/K is unramified, the claim is obvious. Assume that the field K_1 is totally ramified over K, so that the residue field k_1 of the local ring O_1 coincides with k. Let $f \in O_1$ be the uniformizing element in the discrete vaulation ring O_1. Since the field k is of characteristic $\mathrm{char}\, k = 0$, the Galois group G is cyclic, say $G = \mathbb{Z}/l\mathbb{Z}$. Moreover, $O_1 = k((f))$, $f^l \in O \subset O_1$ uniformizes O, and the module $\Omega^1(O_1/\mathbb{C})$ of Kähler differentials splits canonically into the direct sum

$$
\Omega^1(O_1/\mathbb{C}) = \left(\Omega^1(k/\mathbb{C}) \otimes_k O_1 \right) \oplus O_1 \cdot df.
$$

Therefore the module $\Omega^n(O_1/\mathbb{C})$ splits into the direct sum

$$
\Omega^n(O_1/\mathbb{C}) = \left(\Omega^n(k/\mathbb{C}) \otimes_k O_1 \right) \oplus \left(\Omega^{n-1}(k/\mathbb{C}) \otimes_k O_1 \cdot df \right).
$$

The inclusion $\Omega^n(O/\mathbb{C}) \subset \Omega^n(O_1/\mathbb{C})^G$ is obvious. Moreover, since $O_1^G = O$ and $\Omega^i(k/\mathbb{C})$ is G-invariant for every i, we have the inverse inclusion

$$
(\Omega^n(k/\mathbb{C}) \otimes_k O_1)^G = \Omega^n(k/\mathbb{C}) \otimes_k O \subset \Omega^n(O/\mathbb{C}).
$$

It remains to prove the inclusion

$$\left(\Omega^{n-1}(k/\mathbb{C}) \otimes_k O_1 \cdot df\right)^G \subset \Omega^{n-1}(O/\mathbb{C}).$$

Since $O_1 = k[[f]]$, we have

$$\Omega^{n-1}(k/\mathbb{C}) \otimes_k O_1 \cdot df = \prod_{p \geq 0} \Omega^{n-1}(k/\mathbb{C}) \cdot f^p df.$$

To finish the proof, we note that a 1-form $f^p df$ is G-invariant if and only if $p + 1 = l(q + 1)$ for some integer q, and in this case we have

$$\Omega^{n-1}(k/\mathbb{C}) \cdot f^p df = \Omega^{n-1}(k/\mathbb{C}) \cdot x^q dx \subset \Omega^{n-1}(O/\mathbb{C}). \qquad \square$$

Assume now given a smooth crepant resolution $\pi : \widetilde{X} \to X$ of a symplectic quotient singularity $X = V/G$ which is equipped with the holomorphic symplectic form Ω provided by Proposition 3.2. We will need the following fundamental fact.

Proposition 3.5. *Let Y be a smooth algebraic variety equipped with maps $\pi_1 : Y \to V$ and $\pi_2 : Y \to \widetilde{X}$ so that the square*

(3.1)
$$\begin{array}{ccc} Y & \xrightarrow{\pi_2} & \widetilde{X} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{\sigma} & X \end{array}$$

is commutative. Then we have

$$\pi_1^* \Omega_V = \pi_2^* \Omega,$$

where Ω_V is the canonical symplectic form on V, and Ω is the canonical symplectic form on $\widetilde{X}$.

Proof.[3] Since Y is smooth, it suffices to prove the claim after replacing Y with its generic point $y \in Y$. Moreover, we can assume that $\pi_2(y) \notin \pi^{-1}(X_0) \subset \widetilde{X}$, since otherwise the claim follows from definition.

Heuristically, the idea of the proof is to embed y as a special point $y \in Y$ into a smooth curve $Y \subset \widetilde{X}$ which satisfies the conditions of the proposition, and whose generic point lies in $\pi^{-1}(X_0) \subset \widetilde{X}$.

[3]A different proof based on the same general idea can be found in [V1, Proposition 4.5].

Let $x = \pi_2(y) \subset \widetilde{X}$ be the image of the point y. Consider the local ring $\mathcal{O}_{\widetilde{X},x}$ of functions near x, and let

$$U = \operatorname{Spec} \mathcal{O}_{\widetilde{X},x} \subset \widetilde{X}$$

be the open neighborhood of $x \subset \widetilde{X}$. The intersection $U \cap \widetilde{X}_0$ is not empty, since it contains the generic point of $\widetilde{X}$. Take an arbitrary non-trivial function f on U which vanishes on $U \cap (\widetilde{X} \setminus \widetilde{X}_0)$, and choose a point $z \in U$ in the generic fiber of the map $f : U \to \mathbb{A}^1$ which is closed in this generic fiber. Then the field of definition of the point z is of degree of transcedence $(\dim Y + 1)$ over $\mathbb{C}$, and the closure $Z = \overline{z} \subset U$ is an open curve in U with two points: the generic point $z \in Z$ and the special point $y \in Z$.

Consider the pullback $\widetilde{Z} = Z \times_X V$, and let $\widehat{Z}$ be the normalization of the curve $\widetilde{Z}$. The curve $\widehat{Z}$ is a smooth open curve equipped with canonical maps $\pi_{Z,1} : \widehat{Z} \to V$ and $\pi_{Z,2} : \widehat{Z} \to \widetilde{X}$ which satisfy the conditions of the propositon. Moreover, the projection π_2 maps every generic point of the curve $\widehat{Z}$ into the open subset $\widetilde{X}_0 \subset \widetilde{X}$. Therefore we know that

$$(3.2) \qquad\qquad \pi_{Z,1}^* \Omega_V = \pi_{Z,2}^* \Omega.$$

Now, since the square (3.1) is commutative, the maps $\pi_1 : y \to V$, $\pi_2 : y \to Z \subset \widetilde{X}$ both factor through a map $\nu : y \to \widetilde{Z} = V \times_X Z$. Therefore we can replace y with its image $\widetilde{y} = \nu(y) \subset \widetilde{Z}$, which is a closed point in $\widetilde{Z}$.

Choose a closed point $\widehat{y} \in \widehat{Z}$ étale over $\widetilde{y} \subset \widetilde{Z}$, and let $\tau : \widehat{y} \to \widetilde{y}$ be the projection. By construction we have $\pi_{Z,1}|_{\widehat{y}} = \pi_1 \circ \tau$ and $\pi_{Z,2}|_{\widehat{y}} = \pi_2 \circ \tau$, so that (3.2) yields

$$\tau^* \pi_1^* \Omega_V = \tau^* \pi_2^* \Omega.$$

Since $\tau : \widehat{y} \to \widetilde{y}$ is étale, this implies the claim. $\qquad\square$

We end this section with the following corollary of Lemma 3.3, which is a reformulation of Theorem 1.3 (i).

Corollary 3.6. *Let $\mathfrak{t}_V$ be an arbitrary G-invariant vector field on the symplectic complex vector space V, and let $\mathfrak{t}_0$ be the induced vector field on the open subset $X_0 \subset X$ of the quotient variety $X = V/G$.*

Then for every smooth crepant resolution $\pi : \widetilde{X} \to X$, the vector field $\mathfrak{t}_0$ lifts to a vector field $\mathfrak{t}$ on the whole smooth algebraic variety X.

Proof. The holomorphic symplectic form Ω induces an isomorphism

$$\Omega^1(X_0) \cong \mathcal{T}(X_0)$$

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between the sheaf $\Omega^1(X_0)$ of 1-forms and the sheaf $\mathcal{T}(X_0)$ of vector fields, and a compatible isomorphism $\Omega^1(\widetilde{X}) \cong \mathcal{T}(\widetilde{X})$. Let $\alpha \in H^0(X_0, \Omega^1(X_0))$ be the 1-form corresponding to the vector field $\mathfrak{t}_0 \in H^0(X_0, \mathcal{T}(X_0))$. It suffices to prove that the pullback $\pi^*\alpha$ extends to the whole $\widetilde{X}$. This follows from the assumption and from Lemma 3.3. $\hfill\square$

Remark 3.7. In fact, an analogous result holds in the more general situation of Calabi-Yau manifolds. More precisely, instead of symplectic vector spaces one can consider vector spaces equipped with a volume form. If the group G preserves the volume form, then Corollary 3.6 immediately extends to crepant resolutions of the quotient $X = V/G$. The proof is the same, with one modification – one has to identify vector fields with $(n-1)$-forms rather than with 1-forms.

4 Stratification by rank

As in the last section, let V be a complex vector space equipped with a non-degenerate symplectic form Ω_V, and let $G \subset \mathrm{Aut}(V)$ be a finite group of automorphisms of the vector space V which preserves the form Ω_V. We will now introduce a canonical stratification on the singular quotient variety $X = V/G$, which we will call *stratification by rank*.

To do this, for every vector $v \in V$ let $G_v \subset G$ be the stabilizer subgroup of the vector v, that is, the subgroup of elements $g \in G$ such that $g \cdot v = v$. Define the *rank* $\mathsf{rk}\, v$ of the vector $v \in V$ as one half of the codimension

$$\mathsf{rk}\, v = \frac{1}{2}\left(\mathsf{dim}\, V - \mathsf{dim}\, V^{G_v}\right)$$

of the subspace $V^{G_v} \subset V$ of G_v-invariant vectors. The number $\mathsf{rk}\, v$ obviously depends only on the image $\sigma(v) \in X$ of the vector $v \in V$ under the quotient map $\sigma : V \to X = V/G$.

Note that the restriction of the symplectic form Ω_v to the subspace $V \subset V^{G_v}$ is necessarily non-degenerate. Therefore $\mathsf{dim}\, V^{G_v}$ is always even, and $\mathsf{rk}\, v$ is an integer for every vector $v \in V$.

For every integer $k \geq 0$, let $X_k \subset X$ be the subset of points $x \in X$ such that the corresponding G-orbit in V consists of vectors of rank k. The subsets X_k are locally closed subvarieties in X, and $X_p \subset \overline{X_q}$ implies $p \geq q$. The largest subset $X_0 \subset X$ is open and dense. The same is true for the unions

$$X_{\leq p} = \bigcup_{q \leq p} X_q.$$

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Lemma 4.1. *The subset $X_k \subset X$ is a (non-connected) smooth algebraic variety of dimension $\dim Y_k = \dim X - 2k$. Moreover, the projection map $\sigma^{-1}(X_k) \to X_k$ is étale.*

Proof. Let $V_k = \sigma^{-1}(X_k) \subset V$. Consider a vector $v \in V_k$ and let $x = \sigma(v) \in X_k$. Let $\widehat{V}$ be the formal completion of V at v, and let $\widehat{X}$ be the formal completion of X at x. The projection map $\sigma : \widehat{V} \to \widehat{X}$ induces an isomorphism $\widehat{V}/G_v \cong \widehat{X}$. Moreover, it identifies the formal completion $\widehat{X}_k \subset \widehat{X}$ of the subvariety $X_k \subset X$ at x with the quotient $\widehat{V}_k/G_v$ of formal completion $\widehat{V}_k \subset \widehat{V}$ of the subvariety $V_k \subset V$ at $v \in V_k$.

But the completion $\widehat{V}_k$ is isomorphic to the completion at 0 of the vector subspace $V^{G_v} \subset V$ of G_v-invariant vectors. Therefore G_v acts trivially on $\widehat{V}_k$, the quotient $\widehat{V}_k/G_v = \widehat{V}_k$ is smooth and the quotient map $\sigma : V_k \to X_k$ is étale at v.

The dimension formula follows directly from the definition of the rank $\operatorname{rk} v$. $\qquad\square$

As a consequence of this lemma, we see that the G-invariant symplectic form Ω on the vector space V defines a 2-form Ω_k on every stratum X_k. Since the restriction $\Omega|_{V^{G_v}}$ is non-degenerate for every $G_v \subset G$, all these forms are non-degenerate holomorphic symplectic forms.

Let $Y \subset X_k$ be a connected component of the locally closed subvariety $X_k \subset X$. While for $k \geq 1$ the subvariety $Y \subset X$ lies in the singular locus of X, it still admits a sort of a "tubular neighborhood". Namely, let $\widehat{Y}$ be the formal completion of X along $Y \subset X$. Moreover, choose a connected component $V_Y \subset \sigma^{-1}(Y)$ of the preimage $\sigma^{-1}(Y) \subset V$ of $Y \subset X$ under the quotient map $\sigma : V \to X$, and let $\widehat{V}_Y$ be the formal completion of V along $V_Y \subset V$.

Lemma 4.2. (i) *The stabilizer subgroup $G_v \subset G$ is the same for every vector $v \in V_Y$.*

(ii) *The quotient map $\sigma : V \to X$ induces an étale map*
$$\widehat{V}_Y/G_v \to \widehat{Y}.$$

(iii) *Let $V' \subset V$ be the G_v-invariant complement to the subspace $V^{G_v} \subset V$ of vectors fixed by G_v, and let $X' = V'/G_v$ be the quotient variety. There exists a canonical direct product decomposition*
$$\widehat{V}_Y \cong \widehat{X'} \times V_Y,$$
where $\widehat{X'}$ is the formal completion of X' along $0 \in X'$.

Proof. (i) and (ii) are clear. The direct product decomposition in (iii) is induced by the the direct sum decomposition $V = V' \oplus V^{G_v}$. $\qquad\square$

It is easy to give a combinatorial description of the set of components Y of every stratum X, but we will do it only for the simplest case and under the restrictive assumptions of Theorem 1.9. It is the only case which we will need later on.

Lemma 4.3. *Let $V_{\mathbb{R}}$ be a real vector vector space equipped with an action of a finite group G. Let $V_o = V_{\mathbb{R}} \otimes \mathbb{C}$, let $V = V_o \oplus V_o$, and let $X = V/G$ be the quotient of the vector space V by the natural G-action.*

Then the connected components $Y \subset X$ os the first stratum $X_1 \subset X$ of the stratification by rank are in a natural one-to-one correspondence with the conjugacy classes of reflections in the subgroup $G \subset V_o$. Moreover, every reflection in $G \subset V_o$ is of order 2.

Proof. First of all, every reflection $g \in G \subset \operatorname{Aut} V_o$ preserves the real subspace $V_{\mathbb{R}}$. Therefore it acts by multiplication by a *real* scalar, that is, ± 1. Thus g is of order two.

Let $g \in G \subset \operatorname{Aut} V_o$ be a reflection. Consider the subspace $V^g \subset V$ of g-invariant vectors and the image $\sigma(V^g) \subset X$ under the quotient map $\sigma : V \to X$. The generic vector $v \in V^g$ does not lie in any smaller G-invariant subspace in V. Therefore $\operatorname{rk} v = 1$, and the intersection $Y^g = \sigma(V^g) \cap X_1$ is dense in $\sigma(V^g)$.

Since $\sigma(V^g)$ is irreducible, the intersection Y^g is a connected component of the stratum X_1. Moreover, for two reflections $g_1, g_2 \in G$ we have $Y^{g_1} = Y^{g_2} \subset X_1$ if and only if the reflections g_1 and g_2 are conjugate within G. This shows that the correspondence $g \mapsto Y^g$ is injective.

Finally, for every connected component $Y \subset X_1$ let $v \in V$ be an arbitrary vector in the preimage $\sigma^{-1}(Y) \subset V$. Then $\operatorname{rk} v = \operatorname{codim} V_{\mathbb{R}}^{G_v} = 1$, and the stabilizer subgroup $G_v \subset G$ is in fact a subgroup in the group $\operatorname{Aut}(V_{\mathbb{R}}/V_{\mathbb{R}}^{G_v}) = \mathbb{R}^*$. Therefore G_v contains a single non-trivial elementr $g \neq 1 \in G_v = G$, the element $g \in G \subset \operatorname{Aut} V_o$ is a reflection, and we have $Y = Y^g$. Since Y is arbitrary, this implies that the correspondence $g \mapsto Y^g$ is one-to-one. $\qquad\square$

Assume now given a smooth crepant resolution $\pi : \widetilde{X} \to X$ of the quotient variety X. The stratification X_k induces a locally closed stratification

$$\widetilde{X}_k = \pi^{-1}(X_k) \subset \widetilde{X}.$$

The strata $\widetilde{X}_k$ are no longer necessarily smooth.

As in [V1, Proposition 4.16], Proposition 3.5 immediately implies the following.

Proposition 4.4. *For every stratum $\widetilde{X}_k \subset \widetilde{X}$ we have*

$$\dim \widetilde{X}_k \leq \dim_X -k.$$

Proof. Let $Y \subset \widetilde{X}_k$ be the non-singular part of the stratum X_k. Let $y \in Y$ be an arbitrary point, let $T_y Y \subset T_y \widetilde{X}$ be the tangent space to Y at the point y, and let $T_{vert} \subset T_y Y$ be the kernel of the differential $d\pi : T_y Y \to T_{\pi(y)} X_k$. By Proposition 3.5, the restriction of the canonical non-degenerate 2-form Ω to the subvariety $Y \subset \widetilde{X}$ coincides with the pullback of the canonical non-degenerate 2-form on the stratum X_k. Therefore T_{vert} is orthogonal to $T_y Y$ with respect to the form Ω. This implies that

$$\dim T_{vert} + \dim T_y Y \leq \dim T_y \widetilde{X}.$$

But $\dim T_y Y = \dim Y$, $\dim T_y \widetilde{X} = \dim X$ and $\dim T_{vert} = \dim Y - \dim X_k$. Therefore

$$2 \dim Y \leq \dim X + \dim X_k = 2 \dim X - 2k,$$

which proves the claim. □

In particular, we see that the union

$$\widetilde{X}_{\leq 1} = \widetilde{X}_0 \cup \widetilde{X}_1 \subset \widetilde{X}$$

is an open subset whose complement is of codimension > 1. This fact will be crucial for all our constructions.

Remark 4.5. Later (Proposition 5.2) we shall see that for every k the stratum $\widetilde{X}_k \subset \widetilde{X}$ is equidimensional over the stratum $X_k \subset X$. Thus Proposition 4.4 means that the resolution $\pi : \widetilde{X} \to X$ is semismall.

5 The Picard group

We will now use the rank stratification to study resolutions of the symplectic quotient variety $X = V/G$. We begin with the following general observation.

Lemma 5.1. *The Picard group $\mathrm{Pic}(X_0)$ of the non-singular open stratum $X_0 \subset X$ is a torsion group. Therefore, every projective resolution $\pi : \widetilde{X} \to X$ is canonically isomorphic to the blow-up*

$$\mathrm{Bl}\,(X, \mathcal{E}) \to X$$

of a sheaf $\mathcal{E} \subset \mathcal{O}_X$ of ideals on X.

Proof. The quotient map $\sigma : V \to X$ is étale over $X_0 \subset X$, so that modulo torsion $\mathrm{Pic}(X_0)$ is a subgroup in $V_0 = \sigma^{-1}(X_0) \subset V$. But the complement to V_0 in V is codimension > 1 by Lemma 4.1. Therefore $\mathrm{Pic}(V_0) = \mathrm{Pic}(V) = 0$. This proves the first claim.

To prove the second claim, let L be a line bundle on $\widetilde{X}$ which is very ample for the map π, so that

$$\widetilde{X} = \mathrm{Proj} \bigoplus_{k \geq 0} \pi_*(L^k).$$

The projective resolution $\pi : \widetilde{X} \to X$ is by definition one-to-one over an open dense subset $U \subset X$. Since X is normal, we can assume that the complement $X \setminus U \subset X$ is of codimension $\mathsf{codim}\, X \setminus U \geq 2$.

Replace U with $U \cap X_0$, so that $\mathrm{Pic}(U) = \mathrm{Pic}(X_0)$, and denote by $\widetilde{U} = \pi^{-1}(U) \subset \widetilde{X}$ the preimage of the open subset $U \subset X$. Note that since the subset $X \setminus X_0 \subset X$ is of codimension $\mathsf{codim}\, X \setminus X_0 \geq 2$, we still have $\mathsf{codim}\, X \setminus U \geq 2$.

By assumption $\widetilde{U} \cong U$ and $\mathrm{Pic}(\widetilde{U}) = \mathrm{Pic}(U) = \mathrm{Pic}(X_0)$ is a torsion group. Replacing the line bundle L with its positive power, we can assume that it is trivial on $\widetilde{U}$.

Choose a trivialization map $L \to \mathcal{O}_{\widetilde{U}}$, or, equivalently, $\pi_* L \to \mathcal{O}_U$. Since the complement to U in the normal variety X is of codimension ≥ 2, this trivialization map extends to an embedding

$$\pi_* L \to \mathcal{O}_X.$$

Denoting $\mathcal{E} = \pi_* L \subset \mathcal{O}_X$, we get the second claim. $\qquad\square$

Let now $\pi : \widetilde{X} \to X$ be a smooth projective crepant resolution of the quotient variety $X = V/G$. Lemma 5.1 allows us to prove the first of the results announced in Section 1.

Proof of Theorem 1.3. (i) is Corollary 3.6. To prove (ii), note that the Lie algebra $\mathfrak{h}$ of the group H acts naturally on the resolution $\widetilde{X}$ by (i). Therefore the ideal $\mathcal{E} \subset \mathcal{O}_X$ provided by Lemma 5.1 is $\mathfrak{h}$-invariant. Since the group H is by assumption connected, the ideal $\mathcal{E} \subset \mathcal{O}_x$ is also H-invariant, which implies that the H-acton lifts to the resolution $\widetilde{X} = \mathrm{Bl}(X, \mathcal{E})$.

To derive (iii) from (ii), it suffices to notice that the algebraic group $GL(V, G)$ of G-equivariant linear automorphisms of the vector space V is connected. $\qquad\square$

Next, we note that all the results used to establish Theorem 1.3, in particular, Lemma 5.1 and Corollary 3.6, carry over literally to the formal

setting of Theorem 1.4, with the same proofs. We will use this to show that Theorem 1.4 is also a corollary of Lemma 5.1.

Proof of Theorem 1.4. Assume given a projective smooth formal crepant resolution $\widehat{\pi} : \mathfrak{X} \to \widehat{X}$ of the completion of the quotient variety $X = V/G$ at $0 \in X$. Let $\widehat{\mathcal{E}} \subset \widehat{\mathcal{O}}_X$ be the ideal such that $\mathfrak{X} = \mathrm{Bl}(\widehat{X}, \widehat{\mathcal{E}})$.

Let the group $\mathbb{C}^*$ act on the vector field V by homoteties. This action commutes with the G-action and defines a $\mathbb{C}^*$-action on the quotient V/G or, equivalently, a grading on the algebra $\mathcal{O}_X$ of functions on X. This grading induces a decreasing filtration on the completion $\widehat{\mathcal{O}}_X$, and the associated graded quotient with respect to this filtration is the algebra $\mathcal{O}_X$. Let $\mathcal{E} \subset \mathcal{O}_X$ be the associated graded quotient to the ideal $\widehat{\mathcal{E}} \subset \widehat{\mathcal{O}}_X$.

The differential of the $\mathbb{C}^*$-action on V, namely, the Euler vector field ξ_V on V, induces a vector field ξ on the quotient $X = V/G$ and on the completion $\widehat{X}$. By Corollary 3.6, the vector field ξ preserves the ideal $\widehat{\mathcal{E}} \subset \widehat{\mathcal{O}}_X$. Therefore this ideal is isomorphic to the completion of its associated graded quotient $\mathcal{E} \subset \mathcal{O}_X$. Let

$$\widetilde{X} = \mathrm{Bl}\,(X, \mathcal{E})$$

be the blow-up of the ideal $\mathcal{E} \subset \mathcal{O}_X$, and let $\pi : \widetilde{X} \to X$ be the projection.

By construction the map π is projective. Moreover, since $\widehat{\mathcal{E}} \subset \widehat{\mathcal{O}}_X$ is the completion of $\mathcal{E} \subset \mathcal{O}_X$, the completion of $\pi : \widetilde{X} \to X$ along $\pi^{-1}(0) \subset X$ coincides with $\widehat{\pi} : \mathfrak{X} \to X$. Therefore $\widetilde{X}$ is smooth in an open neighborhood $U \subset X$ of $\pi^{-1}(0)$, and the canonical bundle K_U is trivial.

But the ideal $\mathcal{E} \subset \mathcal{O}_X$ is by construction homogenous with respect to the grading given by the Euler vector field. Therefore the $\mathbb{C}^*$-action on $X = V/G$ lifts to a $\mathbb{C}^*$-action on $\widetilde{X}$. Since for every point $x \in \widetilde{X}$ we have

$$\lim_{\substack{\lambda \to 0 \\ \lambda \in \mathbb{C}^*}} \pi(\lambda \cdot x) = 0,$$

we have $\lambda \cdot x \in U \subset \widetilde{X}$ for some $\lambda \in \mathbb{C}^*$, and this implies that $\widetilde{X}$ is smooth everywhere. Moreover, since the canonical bundle $K_{\widetilde{X}}$ is $\mathbb{C}^*$-equivariant, and the canonical bundle K_U is trivial, the whole bundle $K_{\widetilde{X}}$ is trivial, and the resolution $\pi : \widetilde{X} \to X$ is crepant. $\square$

Consider now the rank stratification of the variety X, and let $Y \subset X_k$ be a connected component of an arbitrary stratum $X_k \subset X$. We will use Lemma 5.1 to describe the structure of the resolution $\pi : \widetilde{X} \to X$ near the subvariety $Y \subset X$.

To do this, return to the setting of Lemma 4.2. Denote by $\widehat{Y}$ the formal completion of X near the subvariety $Y \subset X$. Choose a connected component $V_Y \subset \sigma^{-1}(Y)$ of the preimage $\sigma^{-1}(Y) \subset V$ of the subvariety $Y \subset X$ under the quotient map $V \to X$, and let $\widehat{V}_Y$ be the completion of V along $V_Y \subset V$.

Choose an arbitrary vector $v \in V_Y$, let $G_v \subset G$ be the stabilizer subgroup of the vector v, and let $V' \subset V$ be the unique G_v-invariant complement to the subspace $V^{G_v} \subset V$ of G_v-invariant vectors.

By Lemma 4.2, we have a canonical étale map

$$\sigma : \widehat{V}_Y \to \widehat{Y}.$$

Moreover, the quotient $X' = V'/G_v$ does not depend on the choice of the point $v \in V_Y$, and we have a direct product decomposition

$$\widehat{V}_Y \cong \widehat{X'} \times V_Y,$$

where $\widehat{X'}$ is the completion of the quotient X' near $0 \in X'$.

Let $\mathcal{Y}_0$ be the completion of $\widetilde{X}$ along the subvariety $\widetilde{Y} = \pi^{-1}(Y) \subset \widetilde{X}$, and let $\mathcal{Y}$ be the fibered product given by the diagram

$$
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{Y}_0 \\
{\scriptstyle \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
\widehat{V}_Y/G_v & \xrightarrow{\ \sigma\ } & \widehat{Y}.
\end{array}
$$

Proposition 5.2. *There exists a a formal smooth projective crepant resolution $\pi' : \mathfrak{X}' \to \widehat{X'}$ of the quotient variety $X' = V'/G_v$ and a direct product decomposition*

(5.1) $$\mathcal{Y} \cong \mathfrak{X}' \times V_Y,$$

such that

$$\pi = \pi' \times \mathsf{id} : \mathcal{Y} \cong \mathfrak{X}' \times V_Y \to \widehat{V}_Y/G_v \cong \widehat{X'} \times V_Y.$$

Proof. Since the map $\sigma : V_Y \times \widehat{X'} \to \widehat{Y}$ is étale, Lemma 5.1 implies that

$$\mathcal{Y} = \mathrm{Bl}\left(V_Y \times \widehat{X'}, \mathcal{E}\right),$$

where $\mathcal{E}$ is a sheaf of ideals on $V_Y \times \widehat{X'}$. Let $\pi_1 : \widehat{V}_Y \cong \widehat{X'} \times V_Y \to \widehat{X'}$ be the projection onto the first factor. To obtain the direct product decomposition (5.1), it suffices to prove that $\mathcal{E} = \pi_1^* \mathcal{E}'$ for some sheaf of ideals $\mathcal{E}'$ on $\widehat{X'}$.

Every vector $v \in V$ defines a constant vector field $\mathbf{t}_v$ on the vector space V. If the vector $v \in V^{G_v}$ is G_v-invariant, then the vector field $\mathbf{t}_v$ is also G_v-invariant and descends to a vector field on the quotient V/G_v. Completing along $V_Y \subset V/G_v$, we obtain a vector field $\mathbf{t}_v$ on $\widehat{V}_Y \cong \widehat{X'} \times V_Y$.

Vector fields $\mathbf{t}_v$, $v \in V^{G_v}$ are parallel to the fibers of the projection $\pi_1 : \widehat{V}_Y/G_V \to \widehat{X'}$ and generate the relative tangent bundle of $\widehat{V}_Y/G_v$ over $\widehat{X'}$. Therefore, to prove that the sheaf of ideals $\mathcal{E}$ satifies $\mathcal{E} = \pi_1^* \mathcal{E}'$ for some $\mathcal{E}'$ on $\widehat{X'}$, it suffices to prove that $\mathcal{E}$ is preserved by the vector field $\mathbf{t}_v$ for an arbitrary $v \in V^{G_v}$. This is equivalent to proving that the vector field $\mathbf{t}_v$ lifts to a vector filed on the resolution $\mathcal{Y}$ of the variety $\widehat{X'} \times V_Y$. But this follows from Lemma 3.3 by exactly the same argument as Corollary 3.6. Thus the ideal $\mathcal{E}$ on $\widehat{X'} \times V_Y$ is preserved by all the vector fields $\mathbf{t}_v$, so that we have $\mathcal{E} = \pi_1^* \mathcal{E}'$ for some ideal $\mathcal{E}'$ on $\widehat{X'}$.

It remains to prove that the resolution

$$ \pi' : \mathfrak{X}' = \mathrm{Bl}\left(\widehat{X'}, \mathcal{E}'\right) \to \widehat{X'} $$

is smooth and crepant. But this immediately follows from the corresponding properties of the resolution $\pi : \mathcal{Y} \to \widehat{X'} \times V_Y$ and from the direct product decomposition (5.1). $\qquad\square$

Our last general result, Theorem 1.6, follows immediately from Theorem 1.4 and Proposition 5.2.

We now restrict the generality and introduce the same assumptions on $\langle V, G \rangle$ as in Theorem 1.9. Namely, assume that the symplectic vector space V is of the form $V = V_o \oplus V_o^*$, where $V_o \subset V$ is a G-invariant Lagrangian subspace. Assume additionally that we have $V_o = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for some real vector space $V_{\mathbb{R}}$, and that the G-action on V_o is induced by a G-action on $V_{\mathbb{R}}$.

Under these assumptions, and in the case of strata of small rank, Proposition 5.2 can be made more precise.

Corollary 5.3. *Let $\pi : \widetilde{X} \to X$ be smooth projective crepant resolution of the quotient variety $X = (V_o \oplus V_o^*)/G$.*

(i) *Over the open stratum $X_0 \subset X$, the map $\pi : \widetilde{X} \to X$ is one-to-one.*

(ii) *Let $\widehat{X}_1$ be the completion of X along the stratum $X_1 \subset X$ of codimension 2, and let $\mathfrak{X}_1$ be completion of $\widetilde{X}$ along the preimage $\widetilde{X}_1 = \pi^{-1}(X_1) \subset \widetilde{X}$. Then the resolution $\pi : \mathfrak{X}_1 \to \widehat{X}_1$ is isomorphic to the blow-up of the completion $\widehat{X}_1$ along the closed subvarety $X_1 \subset \widehat{X}_1$.*

Proof. (i) is immediate from Proposition 5.2. (ii) follows from Proposition 5.2 and Proposition 2.1. □

To combine these two particular cases, let

$$X_{\leq 1} = X_0 \cup X_1 \subset X$$

be the open dense subset of point of rank ≤ 1 in X, and let $\widetilde{X}_{\leq 1} \subset \widetilde{X}$ be its preimage in $\widetilde{X}$.

Corollary 5.4. *The restriction $\pi : \widetilde{X}_{\leq 1} \to X_{\leq 1}$ of the resolution $\pi : \widetilde{X} \to X$ to $\widetilde{X}_{\leq 1}$ is canonically isomorphic to the blow-up*

$$\mathrm{Bl}\,(X_{\leq 1}, X_1) \to X_{\leq 1}$$

of the variety $X_{\leq 1}$ in the closed subvariety $X_1 \subset X_{\leq 1}$.

Proof. Lemma 5.1 and Corollary 5.3. □

Let now M be the set of all connected components of the subvariety $X_1 \subset X$, and consider the free $\mathbb{Q}$-vector space $\mathbb{Q}[M]$ generated by the set M. (Note that by Lemma 4.3 the set M coincides with the set of conjugacy classes of reflections in M.) Denote also by $\mathbb{Q}_+[M] \subset \mathbb{Q}[M]$ the subset of linear combinations of elements of the set M with positive coefficients.

Lemma 5.5. *The rational Picard group $\mathrm{Pic}(\widetilde{X}) \otimes \mathbb{Q}$ of the smooth resolution $\widetilde{X}$ is canonically isomorphic to $\mathbb{Q}[M]$, and the isomorphism can be chosen in such a way that a class $[L] \in \mathbb{Q}[M]$ of a line bundle L on $\widetilde{X}$ which is very ample for the map $\pi : \widetilde{X} \to X$ lies in $\mathbb{Q}_+[M] \subset \mathbb{Q}[M]$.*

Proof. By Proposition 4.4, the complement $\widetilde{X} \setminus \widetilde{X}_{\leq 1}$ is of codimension ≥ 2. Therefore we have $\mathrm{Pic}(\widetilde{X}) \cong \mathrm{Pic}(\widetilde{X}_1)$. By Corollary 5.3 (ii) and Proposition 2.1, the preimage $\widetilde{Y} = \pi^{-1}(Y) \subset \widetilde{X}_{\leq 1}$ of every connected component $Y \subset X_1$ of the subvariety $X_1 \subset X_{\leq 1}$ is a smooth divisor in $X_{\leq 1}$ isomorphic to

$$\widetilde{Y} \cong Y \times \mathbb{C}P^1.$$

Now, the preimage $\sigma^{-1}(Y) \subset V$ is an open subset in the union

$$\bigcup_g V^g \subset V$$

of fixed-points subspaces for reflections $g \subset G$ in the conjugacy class corresponding to Y. We can replace this union with a single space V^g and obtain an isomorphism

$$\overline{Y} \cong V^g/N(g) \subset X,$$

where $N(g) \subset G$ is the normalizer subgroup of the subspace $V^g \subset V$. The complement to $\sigma^{-1}(Y) \cap V^g$ in V^g is of codimension ≥ 2, therefore $\mathrm{Pic}(\sigma^{-1}(Y) \cap V^g) = 0$. Since the quotient map $\sigma : V \to X$ is étale over Y by Lemma 4.1, this implies that $\mathrm{Pic}(Y) \otimes \mathbb{Q} = 0$. Therefore $\mathrm{Pic}(\widetilde{Y}) \otimes \mathbb{Q} = \mathrm{Pic}(\mathbb{C}P^1) \otimes \mathbb{Q} = \mathbb{Q}$, and the restriction to $\widetilde{X}_1 \subset \widetilde{X}_{\leq 1}$ defines a map

$$\mathsf{res} : \mathrm{Pic}(\widetilde{X}) \otimes \mathbb{Q} \cong \mathrm{Pic}(\widetilde{X}_{\leq 1}) \otimes \mathbb{Q} \to \bigoplus_{Y \in M} \mathrm{Pic}(\widetilde{Y}) \otimes \mathbb{Q} = \mathbb{Q}[M].$$

The map res is injective. Indeed, for every line bundle L on $\widetilde{X}_{\leq 1}$ with trivial restriction $L|_{\widetilde{X}_1}$ we have $L \cong \pi^* \pi_* L$, which means that the sheaf $\pi_* L$ is a line bundle. This line bundle is in turn isomorphic to

$$\pi_* L \cong j_* j^* \pi_* L,$$

where $j : X_0 \hookrightarrow X_{\leq 1}$ is the embedding map of the open subset X_0. Since $\mathrm{Pic}(X_0)$ is torsion, we can assume that $j^* \pi_* L \cong \mathcal{O}_{X_0}$ is trivial. Therefore $\pi_* L = j_* \mathcal{O}_{X_0} = \mathcal{O}_{\widetilde{X}_1}$ is trivial as well.

But the map res is also surjective. Indeed, the correspondence $Y \mapsto \mathcal{O}(\widetilde{Y})$ defines a map $\mathsf{cl} : \mathbb{Q}[M] \to \mathrm{Pic}(\widetilde{X}_{\leq 1})$, and the composition

$$\mathsf{res} \circ \mathsf{cl} : \mathbb{Q}[M] \to \mathrm{Pic}(\widetilde{X}_{\leq 1}) \to \mathbb{Q}[M]$$

is multiplication by 2.

To prove the last claim of the lemma, it suffices to choose for every $Y \in M$ a relatively very ample line bundles as the generator for the group $\mathrm{Pic}(\widetilde{Y}) \cong \mathbb{Q}$. To fix the isomorphism once and for all, we will take for such a generator the bundle $\mathcal{O}(1)$ on $\widetilde{Y} \cong Y \times \mathbb{C}P^1$. $\qquad\square$

We can now formulate and prove the main result of this section. For every connected component $Y \in M$ of the subvariety $X_1 \subset X$ denote by

$$\mathfrak{m}_Y \subset \mathcal{O}_X$$

the ideal of the closed subvariety $\overline{Y} \subset X$, and for every element $l \in \mathbb{Q}_+[M]$ which is a linear combination of generators with positive even integer coefficients,

$$(5.2) \qquad l = \sum_{Y \in M} 2k_Y[Y] \in \mathbb{Q}[M],$$

denote by $\mathcal{E}_l \subset \mathcal{O}_X$ the intersection ideal

$$(5.3) \qquad \mathcal{E}_l = \bigcap_{Y \in M} \mathfrak{m}_Y^{k_Y}.$$

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Proposition 5.6. *Let $\pi : \widetilde{X} \to X$ be a smooth projective crepant resolution, and let L be a line bundle on X which is very ample for the map π. Assume that the class $\mathsf{cl}(L) \subset \mathbb{Q}[M]$ is of the form (5.2). Then the given resolution $\pi : \widetilde{X} \to X$ is canonically isomorphic to the blow-up*

$$\mathrm{Bl}\,(X, \mathcal{E}_l) \to X$$

of the ideal $\mathcal{E}_l \subset \mathcal{O}_X$. In particular, if for two smooth crepant resolutions $\pi : \widetilde{X} \to X$, $\pi' : \widetilde{X}' \to X$ with relatively very ample line bundles L, L' the classes $\mathsf{cl}(L), \mathsf{cl}(L') \subset \mathbb{Q}[M]$ are proportional, then the given resolutions $\pi : \widetilde{X} \to X$, $\pi' : \widetilde{X}' \to X$ are isomorphic.

Proof. By Lemma 5.1 the resolution $\pi : \widetilde{X} \to X$ is isomorphic to the blow-up of the ideal $\mathcal{E} = \pi_* L \subset \mathcal{O}_X$. Let $j_1 : X_{\leq 1} \hookrightarrow X$ be the embedding map of the open subset $X_{\leq 1} \subset X$, and let $\widetilde{j}_1 : \widetilde{X}_{\leq 1} \hookrightarrow \widetilde{X}$ be the embedding map of the open subset $\widetilde{X}_{\leq 1} \subset \widetilde{X}$. Since the complement $\widetilde{X} \setminus \widetilde{X}_{\leq 1}$ is of codimension > 1, we have

$$L \cong \widetilde{j}_{1*}\widetilde{j}_1^* L.$$

Therefore

$$\mathcal{E} = \pi_* L = \pi_* \widetilde{j}_{1*}\widetilde{j}_1^* L = j_{1*}\pi_* \widetilde{j}_1^* L.$$

Since $\mathcal{E}_l \cong j_{1*}j_1^* \mathcal{E}_l$, it suffices to prove that $\mathcal{E}_l = \pi_* L \subset \mathcal{O}_{X_{\leq 1}}$ on $X_{\leq 1}$. This follows immediately from Corollary 5.4.

To prove the second claim, it suffices to notice that both classes $\mathsf{cl}(L)$ and $\mathsf{cl}(L')$ can be made equal and of the form (5.2) by multiplication by an appropriate positive integer. This corresponds to replacing L and L' with positive powers, which does not change the resolutions $\pi : \widetilde{X} \to X$ and $\pi' : \widetilde{X}' \to X$. $\qquad\qquad\square$

This immediately yields the following corollary, which is a reformulation of Theorem 1.9.

Corollary 5.7. *If the set M consists of a single element, then every two projective smooth crepant resolutions $\pi : \widetilde{X} \to X$, $\pi' : \widetilde{X}' \to X$ of the quotient variety X are isomorphic.* $\qquad\qquad\square$

Remark 5.8. As we can see from the proof, it is not necessary to require smoothness of the resolutions $\widetilde{X}$, $\widetilde{X}'$ in Proposition 5.6 and Corollary 5.7. It suffices to require the following:

(i) The varieties $\widetilde{X}$, $\widetilde{X}'$ are normal.

(ii) Over $X_{\leq 1} \subset X$, we have

$$\widetilde{X}_{\leq 1} \cong \widetilde{X}'_{\leq 1} \cong \mathrm{Bl}\left(X_{\leq 1}, X_1\right).$$

(iii) The resolutions $\pi : \widetilde{X} \to X$, $\pi' : \widetilde{X}' \to X$ are semismall with respect to the rank stratification on X.

6   The $\mathbb{C}^*$-action

We now turn to the proof of our main result, Theorem 1.7. The proof goes by a rather standard argument and uses a certain $\mathbb{C}^*$-action on the quotient variety $X = V/G$ which has been very well studied in many particular cases (see, e.g., [N1]).

We begin with some generalities. Let Y be a smooth algebraic variety equiped with an algebraic action of the group $\mathbb{C}^*$. For every point $z \in Y$ denote by $\mathbb{C}^* \cdot z \subset Y$ the $\mathbb{C}^*$-orbit of the point x, and let $\overline{\mathbb{C}^* \cdot z} \subset Y$ be its Zariski closure. Choose a point $y \in Y$ fixed under the $\mathbb{C}^*$-action, so that $\mathbb{C}^* \cdot y = \{y\}$. Consider the the subset $S^0_+(y) \subset Y$ defined by

$$S^0_+(y) = \left\{ z \in Y \mid y \in \overline{\mathbb{C}^* \cdot z} \right\},$$

and let

$$S_+(y) = \overline{S^0_+(y)} \subset Y$$

be it Zariski closure in Y. The subvariety $S_+(y) \subset Y$ is called the *attraction subvariety* of the point $y \in Y$.

More generally, for every closed subvariety $Y_0 \subset Y$ consisting of points fixed by $\mathbb{C}^*$, define the attraction subvariety $S_+(Y_0) \subset Y$ by

$$S_+(Y_0) = \overline{\left\{ z \in Y \mid \overline{\mathbb{C}^* \cdot z} \cap Y_0 \neq \emptyset \right\}}.$$

Let $Y_0 \subset Y$ be a connected component of the subvariety $Y_{\mathbb{C}^*} \subset Y$ of points in Y fixed by $\mathbb{C}^*$. Choose an arbitrary point $y \in Y_0$. Let T_yY be the tangent space to Y at the point $y \in Y$, and let

(6.1)
$$T_yY = \bigoplus_{p \in \mathbb{Z}} T^p_y Y$$

be its weight decomposition with respect to the $\mathbb{C}^*$-action: an element $\lambda \in \mathbb{C}^*$ acts on T^p_y by multiplication by λ^p. Recall the following fact.

Lemma 6.1.   (i) *The component $Y_0 \subset Y$ is smooth at $y \subset Y_0 \subset Y$, and the tangent space Y_yY_0 equals*

$$T_yY_0 = T_y^0Y \subset T_yY = \bigoplus_{p \in \mathbb{Z}} T_y^pY.$$

(ii) *The attraction subvariety $S_+(Y_0) \subset Y$ is smooth at $y \subset S_+(Y_0) \subset Y$, and the tangent subspace $T_yS_+(Y_0) \subset T_yY$ equals*

$$T_yS_+(Y_0) = \bigoplus_{p \geq 0} T_y^pY \subset T_yY = \bigoplus_{p \in \mathbb{Z}} T_y^pY.$$

Proof. Consider the formal completion $\widehat{Y}$ of Y at the point $y \in Y$. The group $\mathbb{C}^*$ does not act on the completion $\widehat{Y}$, but the differential of the action is a well-defined vector field ξ on $\widehat{Y}$. The completion $\widehat{Y_0}$ of the component $Y_0 \subset Y$ at $y \subset Y_0$ is the zero set of the vector field ξ. Moreover, the completion $\widehat{S_+(Y_0)}$ of the attraction variety $S_+(Y_0)$ at $y \in S_+(y)$ is a closed subvariety in $\widehat{Y}$, and it is defined by the ideal $I \subset \mathcal{O}_{\widehat{Y}}$ generated by formal function $f \in \mathcal{O}_{\widehat{Y}}$ with $\xi(f) = 0$.

Since the group $\mathbb{C}^*$ is reductive, the completion $\widehat{Y}$, equipped with the vector field ξ, is isomorphic to the completion at 0 of the tangent space T_yY, equipped with the vector field defined by the natural $\mathbb{C}^*$-action on T_yY. Since both completions $\widehat{Y_0}$, $\widehat{S_+(Y_0)}$ depend only on the vector field ξ, it suffices to prove the lemma for T_yY instead of Y. In this setting it is obvious. $\quad\square$

We also note the following obvious functorial property of the attraction varieties.

Lemma 6.2. *If $f : Y \to Z$ is a $\mathbb{C}^*$-equivariant proper algebraic map between two algebraic varieties equipped with $\mathbb{C}^*$-actions, then for any closed subvariety $Y_0 \subset Y$ consisting of points fixed by $\mathbb{C}^*$ we have*

$$f(S_+(Y_0)) \subset S_+(f(Y_0)). \qquad\square$$

Return now to the situation of Theorem 1.7, where $X = V/G$ is the quotient variety of the symplectic vector space

$$V = V_o \oplus V_o^*$$

by the finite group G acting on V_o, and assume given a smooth projective crepant resolution $\pi : \widetilde{X} \to X$. Let the group $\mathbb{C}^*$ act on $V = V_o \oplus V_o^*$ by

$$\lambda \cdot \langle v, v' \rangle = \langle \lambda v, v' \rangle, \qquad \lambda \in \mathbb{C}^*, \quad \langle v, v' \rangle \in V = V_o \oplus V_o^*, \quad v \in V_o, v' \in V_o^*.$$

This action commutes with the action of the finite group G. Consequently, we obtain a $\mathbb{C}^*$-action on the quotient variety $X = V/G$, which by Theorem 1.3 (ii) lifts to a $\mathbb{C}^*$-action on the resolution $\widetilde{X}$. We will call it the *standard* $\mathbb{C}^*$-*action* on a crepant resolution $\pi : \widetilde{X} \to X$.

Note that the symplectic form Ω_V on $V = V_o \oplus V_o^*$ satisfies

$$(6.2) \qquad \qquad \lambda^*(\Omega) = \lambda\Omega$$

for every $\lambda \in \mathbb{C}^*$. By construction the same holds for the symplectic form Ω on the open non-singular part $X_0 \subset X$. The form Ω extends to a non-degenerate holomorphic form Ω on the resolution $\widetilde{X}$ by Proposition 3.2, and the extension also satisfies (6.2) for every $\lambda \in \mathbb{C}^*$.

The crucial part of the proof of Theorem 1.7 is the following fact.

Proposition 6.3. *For every point $x \in X$ fixed by the $\mathbb{C}^*$-action, there exist at most a finite number of points $\widetilde{x} \in \widetilde{X}$ fixed by the $\mathbb{C}^*$-action and such that $\pi(\widetilde{x}) = x$.*

Proof. By Proposition 5.2 and Theorem 1.4, it suffices to prove the claim for the point $0 \in X$. Consider the subvariety $\pi^{-1}(0)_{\mathbb{C}^*} \subset \pi^{-1}(0)$ of points in $\pi^{-1}(0) \subset \widetilde{X}$ fixed by $\mathbb{C}^*$. Since the map $\pi : \widetilde{X} \to X$ is proper, the subvariety $\pi^{-1}(0) \subset \widetilde{X}$ is also proper, and it suffices to prove that $\dim \pi^{-1}(0)_{\mathbb{C}^*} = 0$.

Let $Y \subset \pi^{-1}(0)$ be an irreducible component of the variety $\pi^{-1}(0)_{\mathbb{C}^*}$, and let $y \in Y$ be an arbitrary point in the non-singular part of the variety $Y \subset \pi^{-1}(0)$. Consider the weight decomposition (6.1) at the point $y \in \widetilde{X}$. Note that the form Ω induces a non-degenerate symplectic form on the tangent space $T_y\widetilde{X}$, and the equation (6.2) implies

$$\dim T_y^p\widetilde{X} = \dim T_y^{1-p}\widetilde{X}, \qquad p \in \mathbb{Z}.$$

Consider the attraction subvariety $S_+(Y) \subset \widetilde{X}$ of the subvariety $Y \subset \widetilde{X}$. By Lemma 6.1 we have

$$\dim Y = \dim T_y^0\widetilde{X}$$

and

$$\dim S_+(Y_0) \geq \sum_{p \geq 0} \dim T_y^p\widetilde{X} = \dim Y + \sum_{p > 0} \dim T_y^p\widetilde{X} = \dim Y + n,$$

where

$$2n = \sum_{p > 0} 2 \dim T_y^p\widetilde{X} = \sum_{p > 0} \left(\dim T_y^p\widetilde{X} + \dim T_y^{1-p}\widetilde{X} \right)$$

$$= \sum_p \dim T_y^p\widetilde{X} = \dim \widetilde{X}.$$

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Now, by Lemma 6.2, we have $\pi(S_+(Y)) \subset S_+(0)$, and by the definition of the $\mathbb{C}^*$-action on $V = V_o \oplus V_o^*$ the attraction subvariety $S_+(0) \subset X$ of $0 \in X$ coincides with the quotient

$$S_+(0) = V_o/G \subset X = V/G$$

of the subspace $V_o \subset V$ by the group G. Note that for every subgroup $G_v \subset G$ the intersection $V_o \cap V^{G_v}$ coincides with the invariant vectors subspace $V_o^{G_v} \subset V_o$. Therefore for every k the intersection $S_+(0) \cap X_k$ of the attraction variety $S_+(0) = V_+/G$ with the subvariety $X_k \subset X$ of point of rank k has the dimension

$$\mathsf{dim}\, S_+(0) \cap X_k = \mathsf{dim}\, V_o - k = n - k.$$

By Proposition 4.4, this implies that

$$\mathsf{dim}\, \pi^{-1}(S_+(0)) \leq n.$$

Since $\pi(S_+(Y))$ lies within $S_+(0)$, we conclude that

$$n \geq \mathsf{dim}\, \pi^{-1}(S_+(0)) \geq \mathsf{dim}\, S_+(Y) \geq \mathsf{dim}\, Y + n,$$

which yields the required equality $\mathsf{dim}\, Y = 0$. $\qquad\square$

We can now prove Theorem 1.7. The argument more or less literally repeats the proof of Theorem 4.2 in the paper [KV].

Proof of Theorem 1.7. Consider the subvariety $\widetilde{X}_{\mathbb{C}^*} \subset \widetilde{X}$ of the points fixed by the standard $\mathbb{C}^*$-action on the crepant resolution $\pi : \widetilde{X} \to X$. Since the variety $\widetilde{X}$ is smooth, the subvariety $\widetilde{X}_{\mathbb{C}^*} \subset X$ is a union of smooth connected components.

By definition of the standard $\mathbb{C}^*$-action on X, the quotient $X_o = V_o^*/G \subset X$ of the G-invariant Lagrangian subspace V_o^* by the group G is the subvariety of $\mathbb{C}^*$-fixed point in the quotient $X = V/G$. The generic point of the subvariety $X_o \subset X$ lies in the dense open subset $X_0 \subset X$ of vectors of rank 0. Since the projection $\pi : \widetilde{X} \to X$ is one-to-one over the subset $X_0 \subset X$, there exists a connected component $Y \subset \widetilde{X}_{\mathbb{C}^*}$ of the fixed-points subvariety $\widetilde{X}_{\mathbb{C}^*} \subset \widetilde{X}$ such that the induced map $\pi : Y \to X_o$ is dominant and generically one-to-one.

By Proposition 6.3, the dominant map $\pi : Y \to X_o$ is in fact finite. But the quotient variety $X_o = V_o^*/G$ is normal. Therefore the finite map $\pi : Y \to X_o = V_o^*/G$ is not only generically one-to-one, but induces an isomorphism between Y and X_o. This implies that the quotient $X_o = V_o^*/G$ is smooth.

To finish the proof of Theorem 1.7, it remains to invoke the classic Theorem 1.8. $\qquad\square$

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